Sampled-Data Control Design
A Differential LMI Approach

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Notation

- Real ($\mathbb{R}$), nonnegative real ($\mathbb{R}_+$) and natural numbers ($\mathbb{N}$)
- The trace function is $\text{tr}(\cdot)$
- Positive definite symmetric real matrix $X > 0$
- Maximum eigenvalue $\sigma_{\text{max}}(\cdot)$
- Norm bounded signals - continuous $L_2$ and discrete-time $\ell_2$
- $\psi(t^-) = \lim_{\varepsilon \to 0^-} \psi(t + \varepsilon)$
- $f(t_k) = f(t)|_{t=t_k}$ for $k \in \mathbb{N}$ is denoted $f[k]$
Motivation

- **Stability:** Matrix $A \in \mathbb{R}^{n \times n}$ is

  - Hurwitz stable: $\text{Re}\{\lambda_i(A)\} < 0, \ \forall i = 1, \cdots, n$
  - Schur stable: $|\lambda_i(A)| < 1, \ \forall i = 1, \cdots, n$

\[\Downarrow\]

**Simple and useful relationship:** For any $h > 0$

$A$ Hurwitz stable $\iff e^{Ah}$ Schur stable
Consider $h > 0$ and a polytopic convex set $\mathcal{A}_c \subset \mathbb{R}^{n \times n}$

$$\mathcal{A}_d = \left\{ e^{Ah} : A \in \mathcal{A}_c \right\}$$

**Robust stability (analysis):**

$$\forall A \in \mathcal{A}_c \iff \forall e^{Ah} \in \mathcal{A}_d$$

*Hurwitz* \quad *Schur*
Motivation

- Robust stability (synthesis):
  - Find a matrix $L$ of appropriate dimensions
  
  $$e^{Ah} + \left( \int_0^h e^{A\tau} Bd\tau \right) L \iff \forall (A, B) \in \mathcal{A}_c \times \mathcal{B}_c$$

- Nonlinear and nonconvex parameter dependence
- Hard to solve
- Robust performance similar difficulty!
Motivation

- **Quadratic stability:**

  - Continuous-time polytopic system

    \[ A'S + SA < 0, \quad S > 0, \quad \forall A \in A_c \]

    \[ \downarrow \quad \forall h > 0 \]

  - Sampled-data polytopic system

    \[
    \begin{cases}
    \dot{P}(t) + A'P(t) + P(t)A < 0 & \quad t \in [0, h), \quad \forall A \in A_c \\
    P(0) = P(h) = S > 0
    \end{cases}
    \]

    \[ \uparrow \]

    \[ e^{A'h}Se^{Ah} - S < 0, \quad S > 0, \quad \forall A \in A_c \]
Motivation

- **Differential Linear Matrix Inequality (DLMI):** General form arising in $\mathcal{H}_\infty$ control design

\[
\begin{bmatrix}
\dot{P}(t) + F' P(t) + P(t) F & P(t) J & G' \\
\bullet & -\gamma^2 I & 0 \\
\bullet & \bullet & -I
\end{bmatrix} < 0, \quad t \in [0, h)
\]

\[P(0), \ P(h) \leftarrow LMI\]

- **Linearity** with respect to the parameters
- Feasible solution $P(t)$ of specific form:

  piecewise linear (Allerhand & Shaked, 2013) and (Briat, 2013) polynomial, $\text{sinc}(\cdot)$ functions, ...
Motivation

- **Sampled-data control**: Zero order hold

\[ u \in \mathcal{U} \iff u(t) = u[k], \quad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \mathbb{N} \]

- Sampling times \( t_0 = 0, \quad t_{k+1} - t_k = h > 0, \quad \forall k \in \mathbb{N} \)
- \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) optimization subject to \( u \in \mathcal{U} \)
- Three seminal books:
  
  J. R. Ragazzini & G. F. Franklin, 1958
  T. Chen & B. A. Francis, 1995
  A. Ichicawa & H. Katayama, 2001
Sampled-data control

- **LTI plant**

\[
\dot{x}(t) = Ax(t) + Bu(t) + E_c w_c(t), \quad x(0) = x_0 \\
z(t) = C_c x(t) + D_c u(t)
\]

- \(x(\cdot) \in \mathbb{R}^{n_x}\) is the state
- \(w_c(\cdot) \in \mathbb{R}^{r_c}\) is the exogenous continuous-time input
- \(z(\cdot) \in \mathbb{R}^{n_z}\) is the controlled output
- \(u(\cdot) \in \mathbb{R}^{n_u}\) is the control input

\(u \in \mathcal{U} \iff u(t) = u[k], \; t \in [t_k, t_{k+1}), \; \forall k \in \mathbb{N}\)
Sampled-data control

- State feedback with control noise

\[ u[k] = \hat{C}x[k] + E_d w_d[k - 1] \]

- Dynamic output feedback with measurement noise

\[ y[k] = C_d x[k] + E_d w_d[k - 1] \]

\[ \hat{x}[k] = \hat{A}\hat{x}[k - 1] + \hat{B}y[k] \]
\[ u[k] = \hat{C}\hat{x}[k - 1] + \hat{D}y[k] \]

- Initial condition \( \hat{x}[-1] = 0 \)

- Full order controller \( \hat{x}[\cdot] \in \mathbb{R}^{n_c} \) with \( n_c = n_u + n_x \)!
Sampled-data control

- Sampled-data control design in a general framework

↓

Hybrid Systems

+
Sampled-data control design in a general framework

Hybrid Systems

Bellman’s Principle of Optimality
The closed-loop sampled-data system can be rewritten as a

\[
\dot{\psi}(t) = F\psi(t) + J_c w_c(t)
\]
\[
z(t) = G\psi(t)
\]
\[
\psi(t_k) = H\psi(t^-_k) + J_d w_d[k - 1]
\]

valid in the time interval \( t \in [t_k, t_{k+1}) \) for all \( k \in \mathbb{N} \)

- **Necessary and sufficient condition** for asymptotic stability

\[
\psi(t_k) \to \psi(t^-_{k+1}) \to \psi(t_{k+1}), \quad \forall k \in \mathbb{N}
\]

- **Performance index calculation**
Hybrid systems

- State space matrices

\[ F, J_c, G \]

(open-loop system data)

and

\[ H, J_d \]

(controller state space matrices)

- The process starts at \( t_0 = 0 \) with

\[
\psi(0) = H \psi(0^-) + J_d w_d[\psi_{0^{[-1]}}]
\]
Hybrid systems

Given $\gamma \in \mathbb{R}_+$, consider the function

$$
\rho_\gamma(\psi(0)) = \sup_{w_c \in \mathcal{L}_2, \ w_d \in \ell_2} \|z\|_2^2 - \gamma^2 \left( \frac{\|w_c\|_2^2 + \|w_d\|_2^2}{\text{disturbance}} \right)
$$

\(\mathcal{H}_\infty\) sampled-data control:

$$
J_\infty = \inf \{ \gamma^2 : \rho_\gamma(0) \leq 0 \}
$$

\(\mathcal{H}_2\) sampled-data control:

$$
J_2 = \sum_{\ell=1}^{r_c+r_d} \rho_\infty(\psi_\ell(0))
$$
\( \mathcal{H}_2 \) performance

**Proposition:** Let \( h \in \mathbb{R}_+ \) be given. The hybrid linear system is **asymptotically stable** and the \( \mathcal{H}_2 \) performance index equals the optimal solution to the convex programming problem

\[
J_2 = \inf_{P(\cdot)} \left\{ \text{tr}(J'_c P(h) J_c) + \text{tr}(J'_d P(0) J_d) \right\}
\]

subject to the **DLMI**

\[
\dot{P}(t) + F' P(t) + P(t) F + G' G < 0
\]

satisfying the boundary condition

\[
\begin{bmatrix}
P(h) & H' \\
\bullet & P(0)^{-1}
\end{bmatrix} > 0
\]
\( \mathcal{H}_2 \) performance

- Matrix \( S = P(0) > 0 \) satisfies the **strict Lyapunov inequality**

\[
e^{F'h} H'SH e^{Fh} < S - \int_0^h e^{F't} G'G e^{Ft} dt \geq 0
\]

which admits a solution if and only if \( He^{Fh} \) is **Schur stable**

- The optimal sampled-data \( \mathcal{H}_2 \) control follows from

\[
\inf_{\text{controller}, P(\cdot)} \left\{ \text{tr}(J_c'P(h)J_c) + \text{tr}(J_d'P(0)J_d) \right\} \rightarrow \text{CONVEX}
\]
Proposition: Let \( h \in \mathbb{R}_+ \) be given. The hybrid linear system is asymptotically stable and the \( \mathcal{H}_\infty \) performance index equals the optimal solution to the convex programming problem

\[
J_\infty = \inf_{P(\cdot), \gamma} \gamma^2
\]

subject to the DLMI

\[
\dot{P}(t) + F'P(t) + P(t)F + \gamma^{-2}P(t)J_cJ_c'P(t) + G'G < 0
\]

satisfying the boundary condition

\[
\begin{bmatrix}
P(h) & H' & 0 \\
\bullet & P(0)^{-1} & J_d \\
\bullet & \bullet & \gamma^2 I
\end{bmatrix} > 0
\]

controller
A solution exists provided that $\gamma > 0$ is large enough. 

The optimal sampled-data $\mathcal{H}_\infty$ control follows from

\[
J_\infty = \inf_{\text{controller}, P(\cdot), \gamma} \left\{ \gamma^2 : \begin{bmatrix} P(h) & H' & 0 \\ \bullet & P(0)^{-1} & J_d \\ \bullet & \bullet & \gamma^2 I_{rd} \end{bmatrix} > 0 \right\}
\]

\[
\downarrow
\]

CONVEX
Boundary conditions

- **Aperiodic sampling**

\[
\begin{bmatrix}
P(h) & H' & 0 \\
\bullet & P(0)^{-1} & J_d \\
\bullet & \bullet & \gamma^2 I
\end{bmatrix} > 0, \quad h \in [h_{\text{min}}, h_{\text{max}}]
\]

controller

guaranteed performance (upper bound) optimization!

- **LTI systems**

\[H = I, \quad J_d = 0 \implies P(0) = P(h) = P(t) > 0, \quad \forall t \in [0, h)\]

stationary solution
Linearization

- Four square blocks partitioning, (Scherer, 1995)

\[
P(t) = \begin{bmatrix}
X(t) & V(t) \\
\hat{X}(t) & 
\end{bmatrix}, \quad P(t)^{-1} = \begin{bmatrix}
Y(t) & U(t) \\
\hat{Y}(t) & 
\end{bmatrix}
\]

leads to:

- **Differential LMIs** and boundary LMI constraints
- One-to-one change of variables yields the optimal controller
- Full order controller \( n_c = n_x \) due to pole / zero cancelation!

- Generalization to **Markov jump linear systems**
Numerical issue

Class of convex problems to be solved

\[ f^* = \inf_{(P_0, P_h) \in \Omega} \left\{ f(P_0, P_h) : \mathcal{L}(\dot{P}(t), P(t)) < 0, \; t \in [0, h) \right\} \]

- \( f(\cdot, \cdot) \) is linear
- \( \mathcal{L}(\cdot, \cdot) \) is linear
- \((P(0), P(h)) = (P_0, P_h) \in \Omega \) is expressed by LMIs

\[ P(t) = \sum_{i=0}^{n_\phi-1} X_i \phi_i(t) \]
Numerical issue

- **Piecewise linear:** For \( \eta = h/n_{\phi} \) and \( \phi_i(t) = \phi(t - i\eta) \)

\[
\phi(t) = \begin{cases} 
1 - \frac{|t|}{\eta}, & |t| \leq \eta \\
0, & \text{otherwise}
\end{cases}
\]

- \( P(t) \) is continuous and is feasible if and only if

\[
\mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_i\right) < 0
\]

\[
\mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_{i+1}\right) < 0
\]

\((X_0, X_{n_{\phi}}) \in \Omega\)

Optimal solution in only one shot \( \Leftarrow 2n_{\phi} + 1 \) LMIs!
Numerical issue

- **Polynomial:**
  \[ \phi_i(t) = \left( \frac{t}{h} \right)^i \]

- **Iterative solution by outer linearization**
  - Simple to solve convex (LMI) subproblem
  - Global convergence
  - Optimality test

\[ \max_{t \in [0, h]} \sigma_{\text{max}} \left( \mathcal{L} \left( \dot{P}(t), P(t) \right) \right) < 0 \]
Outer linearization

The diagram shows a graph of $\sigma_{\text{max}}(\cdot)$, which is a function of $t$. The graph is a parabola with its vertex at the point $(1, 0)$. The x-axis is labeled with $t$ and the y-axis is labeled with $\sigma_{\text{max}}(\cdot)$. The scale on the y-axis ranges from -0.5 to 1, and the scale on the x-axis ranges from 0 to 2.
Outer linearization

\[ h(kT) \]

\[ kT \]
Outer linearization

\[ h(\frac{kT}{kT}) \]

\[ \sigma_{\text{max}}(\cdot) \]
Outer linearization

\[ h(kT) \]

\[ kT \left[ \text{ms} \right] \]
Outer linearization
Outer linearization

\[ h(kT) \]

\[ kT \left[ ms \right] \]
Example

Example borrowed from (Ichicawa & Katayama, 2001)

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = E_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C'_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ C_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0 \\ d \end{bmatrix}, \quad E_d = d, \quad h = 1.0 \]

For \( d = 1 \), the optimal \( \mathcal{H}_\infty \) sampled-data controller

\[ C^*(\zeta) = \frac{-0.3225\zeta^3 + 0.8686\zeta^2 - 8.012 \times 10^{-10}\zeta}{\zeta^3 + 0.2865\zeta^2 + 0.01608\zeta + 1.697 \times 10^{-12}} \]

imposes the performance cost \( \gamma_* = 2.16 \).
Example - Piecewise linear

\[ \sqrt{\int_\infty J_\infty} = 1 \]

\[ d = 0 \quad \text{and} \quad d = 1 \]

\[ sdhinf \]
Comparison

- **Piecewise linear**
  - Non iterative method
  - $n_\phi \approx 2^6$ time segments

- **Outer linearization**
  - Iterative method
  - Polynomial function of degree $n_\phi \approx 2 \times 6$
Conclusion

Open problems:

- **Dynamic output feedback control**: in the context of Markov Jump Linear Systems.
- **Nonuniform sampling**: optimality
Conclusion

- This research has been developed in collaboration with

Gabriela Werner Gabriel

Rafael Fernandes Cunha

Tiago Rocha Gonçalves
Conclusion

Thank you for your attention!!!